

The proton-neutron symplectic model of nuclear collective motions

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A proton-neutron symplectic model of collective motions, based on the non-compact symplectic group $Sp(12, R)$, is introduced by considering the symplectic geometry of the two-component many-particle nuclear system. The possible classical collective motions are determined by different dynamical groups that can be constructed from the symplectic generators. The relation of the $Sp(12, R)$ irreps with the shell-model classification of the basis states is considered by extending of the state space to the direct product space of $SU_p(3) \otimes SU_n(3)$ irreps, generalizing in this way the Elliott's $SU(3)$ model for the case of two-component system. The $Sp(12, R)$ model appears then as a natural multi-major-shell extension of the generalized proton-neutron $SU(3)$ scheme which takes into account the core collective excitations of monopole and quadrupole, as well as dipole type associated with the giant resonance vibrational degrees of freedom. Each $Sp(12, R)$ irreducible representation is determined by a symplectic bandhead or an intrinsic $U(6)$ space which can be fixed by the underlying proton-neutron shell-model structure, so the theory becomes completely compatible with the Pauli principle. It is shown that this intrinsic $U(6)$ structure is of vital importance for the appearance of the low-lying collective bands with both the positive and negative parity. The full range of low-lying collective states can then be described by the microscopically based intrinsic $U(6)$ structure, renormalized by coupling to the giant resonance vibrations.

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I. INTRODUCTION

Symmetry is an important concept in physics. In finite many-body systems, it appears as time reversal, parity, and rotational invariance, but also in the form of dynamical symmetries [1]-[5]. The standard symmetry approach allows the construction of a Hamiltonian of a system under consideration which is, or nearly so, invariant under a group of symmetry transformations. Group theory then allows one to construct basis states realizing the symmetry and explicit matrix elements for physically interesting transition operators themselves classified by the symmetry. Many properties of atomic nuclei have been investigated using algebraic models, in which one obtains bands of collective states which span irreducible representations of the corresponding dynamical groups [2],[4],[5].

There is, however, another non-standard approach for exploiting the symmetry by identifying first the generators of possible collective motions and then the algebra they close under commutation [6]-[9]. This reveals directly the physical content of a certain algebraic model. The Hamiltonian of nuclear system is then assumed to be a function of these operators. Along this line, some algebraic models of collective motions in nuclei have been proposed based on the algebras of $SL(3, R)$ [10], $Rot(3)$ [11], $CM(3)$ [12] and $Sp(6, R)$ [13] groups, respectively.

It is known that in formulating the nuclear many-body problem some kinematical requirements should be satisfied by the nuclear wave function [14],[15]. First, the wave function of the nucleus should be realized microscopically, i.e. it should depend upon all single particle variables -spacial and spin variables. Secondly, the nuclear wave function should be translationally-invariant. This means that the wave function of the atomic nu-

cleus, free from external fields, can be expressed as a product of the plane wave, describing the center-of-mass motion, and translationally-invariant wave functions, describing internal properties of the free nucleus. The two conditions can be unified into one and formulated as a requirement of the wave function to be microscopically translationally-invariant. Third, the nuclear wave function should preserve the observed integrals of motion (total angular momentum, its third projection, etc.). An arbitrary wave function fulfilling the above requirements is referred to as a kinematically-correct wave function [14],[15].

In the present paper we exploit further the non-standard symmetry approach to nuclear collective motion and take into account explicitly the proton-neutron degrees of freedom. Hence, we do not use the isospin formalism and formulate the algebraic approach for the two-component nuclear system. This allows to reveal some new features and forms of collective excitations which are missing in the microscopic theory of the one-component nuclear systems. We propose a symplectic model, consistent with the proton-neutron composite (granular) structure of the nucleus and appropriate mainly for the description of different collective excitations in deformed heavy mass even-even nuclei. The consideration of the symplectic geometry shows that the $Sp(12, R)$ group appears as a dynamical group of the collective excitations in the two-component many-body nuclear system.

From the hydrodynamic perspective, the possible classical collective flows are determined by different dynamical groups that can be constructed from the symplectic generators of $Sp(12, R)$. To quantize the model one has to construct the irreducible representations of the $Sp(12, R)$ group, appropriate to the many-particle system. The relation of the $Sp(12, R)$ irreps with the shell-model classification of the basis states is considered by extending of the state space to the direct product space

of $SU_p(3) \otimes SU_n(3)$ irreps, generalizing in this way the Elliott's $SU(3)$ model [16] for the case of two-component system. The $Sp(12, R)$ model appears then as a natural multi-major-shell extension of the generalized proton-neutron $SU(3)$ scheme which takes into account the core collective excitations of monopole and quadrupole, as well as dipole type associated with the giant resonance vibrational degrees of freedom. Each $Sp(12, R)$ irreducible representation is determined by a symplectic bandhead or an intrinsic $U(6)$ space which can be fixed by the underlying proton-neutron shell-model structure, so the theory becomes completely compatible with the Pauli principle. It is shown that this intrinsic $U(6)$ structure is of vital importance for the appearance of the low-lying collective bands with both the positive and negative parity. Then the full range of low-lying collective states can be described by the microscopically based intrinsic $U(6)$ structure, renormalized by coupling to the giant resonance vibrations.

II. THE SYMPLECTIC GEOMETRY

In the microscopic nuclear theory the wave function should depend upon all single particle variables r_1, r_2, \dots, r_A . But in order to avoid the problem of center-of-mass motion it is convenient to use translationally-invariant variables from the very beginning. The set of the translationally-invariant variables is well known and is provided by the set of Jacobi vectors. Additionally, nuclei consist of protons and neutrons. Thus, we consider a two-component nuclear system consisting (after removing of the center-of-mass) of $m = A - 1$ particles and label the set of Jacobi vectors, corresponding to protons and neutrons, by an additional quantum number $\alpha = p, n$ (or $\alpha = 1, 2$). The latter extends the single particle configuration space to \mathbb{R}^6 . Then the Jacobi coordinates $x_{is}(\alpha)$ and corresponding momenta $p_{is}(\alpha)$ of this m particle system, defined by the only nonzero commutator

$$[x_{is}(\alpha), p_{jt}(\beta)] = i\hbar\delta_{ij}\delta_{st}\delta_{\alpha\beta}, \quad (1)$$

where $s, t = 1, 2, \dots, m$, $i, j = 1, 2, 3$, and $\alpha, \beta = p, n$, are the elements of a $6m$ -dimensional Heisenberg-Weyl Lie algebra $hw(6m)$. The Hermitian quadratic expressions in the coordinates and momenta

$$\begin{aligned} & x_{is}(\alpha)x_{jt}(\beta), \\ & x_{is}(\alpha)p_{jt}(\beta) + p_{jt}(\beta)x_{is}(\alpha), \\ & p_{is}(\alpha)p_{jt}(\beta), \end{aligned} \quad (2)$$

close under commutation the symplectic group $Sp(12m, R)$, which is the full dynamical group of the system with $6m$ degrees of freedom.

The problem of $6m$ degrees of freedom defined in the many-particle Hilbert space can be associated with a definite irrep of the dynamical symmetry group $Sp(12m, R)$. The latter is also a dynamical symmetry group of the

$6m$ -dimensional harmonic oscillator that provides a complete set of states for the many-body problem. Clearly, a large class of many-body Hamiltonians can be written in terms of elements in the enveloping algebra of $Sp(12m, R)$. However, it was proved that the collective effects are associated with operators that are scalar in $O(m)$ [15]–[20]. Then, the collective part of the Hamiltonian is obtained by projecting the latter on a definite $O(m)$ irrep [15]–[20] associated with the m Jacobi vectors in the configuration space \mathbb{R}^{6m} , where $m = A - 1$ and A is equal to the total number of nucleons in the system.

The group $Sp(12m, R)$ among its subgroups has

$$Sp(12m, R) \supset Sp(12, R) \otimes O(m), \quad (3)$$

whose generators are obtained from (2) in standard way by means of a contraction. The infinitesimal operators of the $O(m)$ group have the well-known antisymmetrized form

$$L_{st} = \sum_{i,\alpha} \left(x_{is}(\alpha)p_{it}(\alpha) - x_{it}(\alpha)p_{is}(\alpha) \right). \quad (4)$$

For $Sp(12, R)$ there are 78 Hermitian generators which are given by the following one-body operators:

$$Q_{ij}(\alpha, \beta) = \sum_{s=1}^m x_{is}(\alpha)x_{js}(\beta), \quad (5)$$

$$S_{ij}(\alpha, \beta) = \sum_{s=1}^m \left(x_{is}(\alpha)p_{js}(\beta) + p_{is}(\alpha)x_{js}(\beta) \right), \quad (6)$$

$$L_{ij}(\alpha, \beta) = \sum_{s=1}^m \left(x_{is}(\alpha)p_{js}(\beta) - x_{js}(\beta)p_{is}(\alpha) \right), \quad (7)$$

$$T_{ij}(\alpha, \beta) = \sum_{s=1}^m p_{is}(\alpha)p_{js}(\beta). \quad (8)$$

As can be seen from (5)–(8), the $Sp(12, R)$ is generated by those bilinear operators which are invariant under $O(m)$, and hence its generators commute with those of $O(m)$. Thus, the two groups $Sp(12, R)$ and $O(m)$ are complementary [18],[21],[22],[23] within the $Sp(12m, R)$ irrep, i.e. there is a relationship between their irreps. This means that the quantum numbers labeling the $Sp(12, R)$ irrep $\langle \omega \rangle \equiv \langle \omega_1 + \frac{1}{2}m, \dots, \omega_6 + \frac{1}{2}m \rangle$ within the $Sp(12m, R)$ irreps $\langle \frac{1}{2}^{6m} \rangle$ or $\langle \frac{1}{2}^{6m-1} \frac{3}{2} \rangle$ label also the $O(m)$ irrep $(\omega) = (\omega_1, \dots, \omega_6)$ [20],[22]. Due to the complementarity of $Sp(12, R)$ and $O(m)$, the collective states characterized by an $O(m)$ irrep belong to a single irrep of $Sp(12, R)$. The $Sp(12, R)$ group is therefore the dynamical group of collective excitations of the two-component proton-neutron nuclear system. We note that the reduction $Sp(12m, R) \supset Sp(12, R) \otimes O(m)$ is multiplicity free [18],[19],[22],[23].

The set of basis states of the full dynamical symmetry group $Sp(12m, R)$ of the whole many-particle nuclear system contains all possible motions, collective, intrinsic,

etc. However, often, one restricts himself to a certain type of dominating modes in the process under consideration. Thus, by reducing the group $Sp(12m, R)$ one performs the separation of the nuclear variables into kinematical (internal) and dynamical (collective) ones. The choice of the reduction chain depends on the concrete physical problem we want to consider. As we saw, the group $Sp(12, R)$ plays an important role in the treatment of the collective excitations of the proton-neutron nuclear system. The reduction $Sp(12m, R) \supset Sp(12, R) \otimes O(m)$ turns out to be of a crucial importance in the microscopic nuclear theory also because the first group in the direct product subgroup $Sp(12, R) \otimes O(m)$ is associated with the collective excitations, whereas the second group allows one to ensure the proper permutational symmetry of the nuclear wave functions. In this way the considered reduction chain corresponds to the splitting of the microscopic many-particle configuration space \mathbb{R}^{6m} , spanned by the relative Jacobi vectors, into kinematical and dynamical submanifolds, respectively.

It is clear then that a large class of collective Hamiltonians represented by any function of the bilinear combinations of the coordinates and momenta will lie in the enveloping algebra of the $Sp(12, R)$ rather than $Sp(12m, R)$. In particular, the kinetic energy terms for the two subsystems $K(\alpha) = p_\alpha^2/2m_\alpha = \frac{1}{2m_\alpha}T(\alpha, \alpha)$, their harmonic oscillator Hamiltonians $H_0(\alpha) = \frac{1}{2m_\alpha}T(\alpha, \alpha) + \frac{1}{2}m_\alpha\omega_\alpha^2 \sum_{ii} Q_{ii}(\alpha, \alpha)$ are simply elements of the $Sp(12, R)$ algebra, whereas the collective potential, represented usually as a function $v(Q)$ of the mass quadrupole tensor, will be in the enveloping algebra. With the microscopic realization of the mass quadrupole (5), the collective potential $v(Q)$ becomes a well-defined shell-model operator. In this way, the $Sp(12, R)$ algebraic structure embraces both the microscopic collective and the harmonic oscillator shell-model aspects of the nuclear excitations in the proton-neutron many-particle systems.

III. THE DYNAMICAL CONTENT

We consider the dynamical groups of possible collective flows which can be generated by the operators (5)–(8). They play a fundamental role in the algebraic formulation of the nuclear collective motion because they are the groups of collective vibrations and rotations.

In a given collective model, the momentum observables should be the infinitesimal generators of collective flows corresponding to Lie group transformations. The operators $L_{ij}(\alpha, \beta)$ (7) are the infinitesimal generators of rigid rotations in the 6-dimensional space and are the generators of the group $SO(6)$. Among them are the 6 angular momentum components $L_k^{(p)} = \varepsilon_{kij} L_{ij}(p, p)$ and $L_k^{(n)} = \varepsilon_{kij} L_{ij}(n, n)$ ($k, i, j = \text{cyclic}$) of the infinitesimal generators of rigid rotations of the proton and neutron subsystems, respectively. The remaining 9 components of

$L_{ij}(\alpha, \beta)$ with $\alpha \neq \beta$ represent combined proton-neutron collective excitations of the system as a whole.

More general collective flows, e.g. vibrational flows or irrotational flow rotations, are generated by the infinitesimal generators of more general dynamical groups. The shear momentum generators of deformations and rotations $S_{ij}(\alpha, \beta)$ (6) represent the infinitesimal generators of $GL(6, R)$. The six diagonal momenta of them $S_{ii}(p, p)$ and $S_{ii}(n, n)$ are infinitesimal generators of monopole and quadrupole shape vibrations and deformations of the proton and neutron subsystems along the intrinsic axis i , while the off-diagonal components of the shear momenta $S_{ij}(p, p)$ and $S_{ij}(n, n)$ ($i \neq j$) are infinitesimal generators of irrotational-flow rotations of the two-subsystems, respectively. The operators $S_{ii}(p, n)$ and $S_{ii}(n, p)$ represent a simultaneous deformation of the proton and neutron distributions (ellipsoids) along the principal axis i , whereas $S_{ij}(p, n)$ and $S_{ij}(n, p)$ generate irrotational-flow (surface wave) rotations of the combined proton-neutron system.

The operators $S_{ij}(\alpha, \beta)$ together with the angular momenta $L_{ij}(\alpha, \beta)$ close under commutation and span the Lie algebra $gl(6, R)$. In this way, the operators $\{S_{ij}(\alpha, \beta), L_{ij}(\alpha, \beta)\}$ with $\alpha \neq \beta$ extend the direct sum algebra $gl_p(3, R) \oplus gl_n(3, R)$, generated by the set $\{S_{ij}(\alpha, \alpha), L_{ij}(\alpha, \alpha)\}$ with $\alpha = p, n$; $i, j = 1, 2, 3$, to $gl(6, R)$, the algebra of deformations and rotations in a 6-dimensional space, including the excitations of the proton subsystem with respect to the neutron one, as well as excitations of the combined proton-neutron system as a whole.

If we adjoin the quadrupole moments $Q_{ij}(\alpha, \beta)$ to $S_{ij}(\alpha, \beta)$ and $L_{ij}(\alpha, \beta)$, we obtain a basis for the semi-direct sum Lie algebra $gcm(6) = [\mathbb{R}^{21}]gl(6, R)$, the general collective motion algebra in six dimensions. The 21 quadrupole moments $Q_{ij}(\alpha, \beta)$ commute among themselves and span the Abelian Lie algebra \mathbb{R}^{21} . They characterize the shape and the orientation of the proton-neutron nuclear system as a whole in the six-dimensional space, as well the configuration of the proton distribution with respect to the neutron one.

The operators $Q_{ij}(\alpha, \beta)$ and $L_{ij}(\alpha, \beta)$ which are subset of $gcm(6)$ generators span the semi-direct sum Lie algebra $[\mathbb{R}^{21}]so(6)$ which can be denoted as $rot(6)$ by analogy with $rot(3) = [\mathbb{R}^5]so(3)$ algebra. The operators $Q_{ij}(p, p)$ and $L_{ij}(p, p)$ generate $rot_p(3)$ algebra. The $rot_n(3)$ algebra is similarly defined.

Finally, the addition of the 21 momentum operators $T_{ij}(\alpha, \beta)$ to the set of $gcm(6)$ algebra generators extends the latter to the $sp(12, R)$ algebra with total number of 78 generators. It is clear that among the generators of $sp(12, R)$ algebra are the proton and neutron kinetic energy operators.

We note that by contraction with respect to α of the $sp(12, R)$ generators, i.e. $Q_{ij} = \sum_\alpha Q_{ij}(\alpha, \alpha)$, $S_{ij} = \sum_\alpha S_{ij}(\alpha, \alpha)$, $L_{ij} = \sum_\alpha L_{ij}(\alpha, \alpha)$, $T_{ij} = \sum_\alpha T_{ij}(\alpha, \alpha)$, one obtains respectively the generators of the one-component dynamical algebras $rot(3)$, $gl(3, R)$, $gcm(3)$,

and $sp(6, R)$.

It is clear that a wide class of in-phase (isoscalar) and out-of-phase (isovector) excitations of the proton subsystem with respect to the neutron one is present, commonly interpreted in the IBM-2 [25] terms as symmetry and mixed-symmetry states, respectively. In particular, one has the linear and angular collective displacements associated with the giant dipole resonance ($E1$ excitations) and the scissors mode ($M1$ excitations) of the proton system with respect to the neutron one. The latter mode, for example, is generated by the isovector out-of-phase operator $\vec{L} \simeq (\vec{L}^p - \vec{L}^n)$.

The different subgroups of $Sp(12, R)$ reveal some of the possible classical collective flows. Then it appears that the $Sp(12, R)$ group provides a very general framework in which to investigate the nature of collective motions in nuclei. To quantize a given classical collective model one has to construct the irreducible representations of its dynamical symmetry group. This, as will see, could be readily done for the $Sp(12, R)$.

Note that all the algebras considered are fully microscopically realizable, i.e. they are composed of fully microscopic one-body operators which act on the Hilbert space of antisymmetrized many-particle state vectors.

IV. REPRESENTATIONS OF THE $Sp(12, R)$ LIE ALGEBRA

The $Sp(12, R)$ algebra has many nice properties. First, it is a semi-simple Lie algebra with a well-known representation theory. Second, important for the shell-model theory of nuclear collective motion, as we will see, is the fact that its irreps are readily given in shell-model terms. Indeed, the discrete series representations are readily constructed on the many-particle Hilbert state space by the realization of the $Sp(12, R)$ algebra as the vector space of all skew-adjoint one-body bilinear products in the position $x_{is}(\alpha)$ and momentum $p_{is}(\alpha)$ observables. Third, as was shown, the $Sp(12, R)$ algebra contains many of collective motion algebras as subalgebras which reveal the dynamical content of the $Sp(12, R)$ model from the hydrodynamic perspective.

To construct the irreducible representations of the $Sp(12, R)$ Lie algebra in a harmonic oscillator basis, it is convenient to introduce the harmonic oscillator raising and lowering operators

$$\begin{aligned} b_{i\alpha,s}^\dagger &= \sqrt{\frac{m_\alpha\omega}{2\hbar}} \left(x_{is}(\alpha) - \frac{i}{m_\alpha\omega} p_{is}(\alpha) \right), \\ b_{i\alpha,s} &= \sqrt{\frac{m_\alpha\omega}{2\hbar}} \left(x_{is}(\alpha) + \frac{i}{m_\alpha\omega} p_{is}(\alpha) \right), \end{aligned} \quad (9)$$

which satisfy the commutation relations

$$[b_{i\alpha,s}, b_{j\beta,t}^\dagger] = \delta_{ij} \delta_{\alpha\beta} \delta_{st}. \quad (10)$$

In terms of the harmonic oscillator creation and annihilation operators, the many-particle realization of the

$Sp(12, R)$ Lie algebra is given by

$$F_{ij}(\alpha, \beta) = \sum_{s=1}^m b_{i\alpha,s}^\dagger b_{j\beta,s}^\dagger, \quad (11)$$

$$G_{ij}(\alpha, \beta) = \sum_{s=1}^m b_{i\alpha,s} b_{j\beta,s}, \quad (12)$$

$$A_{ij}(\alpha, \beta) = \frac{1}{2} \sum_{s=1}^m (b_{i\alpha,s}^\dagger b_{j\beta,s} + b_{j\beta,s} b_{i\alpha,s}^\dagger). \quad (13)$$

The commutation relation for the $Sp(12, R)$ algebra are easily inferred from the commutation relations (10). The number-conserving operators (13) generate the maximal compact subgroup $U(6)$ of $Sp(12, R)$. We will use also the following notations $F_{ab} \equiv F_{ij}(\alpha, \beta)$, $G_{ab} \equiv G_{ij}(\alpha, \beta)$, $A_{ab} \equiv A_{ij}(\alpha, \beta)$ in which the single indices $a \equiv i\alpha$, $b \equiv j\beta$ are introduced.

In terms of boson operators (11)–(13) the generators (5)–(8) of $Sp(12, R)$ algebra take the form:

$$Q_{ij}(\alpha, \beta) = A_{ij}(\alpha, \beta) + \frac{1}{2} [F_{ij}(\alpha, \beta) + G_{ij}(\alpha, \beta)] \quad (14)$$

$$S_{ij}(\alpha, \beta) = i [F_{ij}(\alpha, \beta) - G_{ij}(\alpha, \beta)], \quad (15)$$

$$L_{ij}(\alpha, \beta) = -i [A_{ij}(\alpha, \beta) - A_{ji}(\beta, \alpha)], \quad (16)$$

$$T_{ij}(\alpha, \beta) = A_{ij}(\alpha, \beta) - \frac{1}{2} [F_{ij}(\alpha, \beta) + G_{ij}(\alpha, \beta)] \quad (17)$$

If we compare the expressions (11)–(13) with the generators (in the coupled angular momentum form) of the phenomenological algebraic Interacting Vector Boson Model [24], it becomes clear that the latter can be considered as *effective* (or renormalized) counterparts of the microscopic many-particle operators, defined by (11)–(13). However, an important difference is that by the operators (11)–(13) we can build generic irreducible representations $E \equiv [E_1, E_2, E_3, E_4, E_5, E_6]$ of $U(6)$ in contrast to the IVBM where only the fully symmetric irreps of $U(6)$ are permitted. As we will see later, this implies some new features which arise in the present approach. We note also that in this respect the labeling of the $U(6)$ irreps in the $Sp(12, R)$ scheme proposed here resembles that of IBM-4 [25].

An $Sp(12, R)$ unitary irreducible representation is characterized by the $U(6)$ quantum numbers $\sigma = [\sigma_1, \dots, \sigma_6]$ of its lowest-weight state $|\sigma\rangle$, i.e. $|\sigma\rangle$ satisfies

$$\begin{aligned} G_{ab}|\sigma\rangle &= 0, \\ A_{ab}|\sigma\rangle &= 0, \quad a < b, \\ A_{aa}|\sigma\rangle &= \sigma_a|\sigma\rangle \end{aligned} \quad (18)$$

for all $a, b = 1, \dots, 6$. Note that the lowest-weight state $|\sigma\rangle$ for a symplectic irrep is also a highest-weight state for the $U(6)$ irrep $[\sigma_1, \dots, \sigma_6]$.

A discrete basis for the irrep $\langle\sigma\rangle \equiv \langle\sigma_1 + \frac{n}{2}, \dots, \sigma_6 + \frac{n}{2}\rangle$ of $Sp(12, R)$ is generated by the repeated application of

the $Sp(12, R)$ two-boson creation operators on this lowest weight state. A classification of the states obtained by this procedure is facilitated by the observation that the raising operators of $Sp(12, R)$ are components of an irreducible tensor of $U(6)$. Indeed, they transform according to the $U(6)$ irreducible representation [2]. Thus by taking tensor products of these raising operators, we define the $U(6)$ tensor operators

$$P^{(n)}(F) = [F \times \dots \times F]^{(n)}, \quad (19)$$

where $n = [n_1, \dots, n_6]$ is a partition with even integer parts. It is known that these couplings are multiplicity free. By a $U(6)$ coupling of these tensor products to the lowest-weight $U(6)$ state $|\sigma\rangle$, one constructs the basis of states for an $Sp(12, R)$ irrep

$$|\Psi(\sigma n \rho E \eta)\rangle = [P^{(n)}(F) \times |\sigma\rangle]_{\eta}^{\rho E}, \quad (20)$$

where $E = [E_1, \dots, E_6]$ indicates the $U(6)$ quantum numbers of the coupled state, η labels a basis of states for the coupled $U(6)$ irrep E and ρ is a multiplicity index. Thus we obtain a basis of $Sp(12, R)$ states that reduce the subgroup chain $Sp(12, R) \subset U(6)$.

TABLE I: The scalar representation $\langle\sigma\rangle = 0$ of $Sp(12, R)$.

...
[8], [62], [44], [422], [2222]
[6], [42], [222]
[4], [22]
[2]
[0]

As an example, the scalar (i.e. $\langle\sigma\rangle = 0$) $Sp(12, R)$ irrep is given in Table I. This example allows us to compare the representation spaces of the present approach with that of IVBM [24]. As can be seen, the representation space of the proton-neutron symplectic model proposed in the present paper even for the scalar irrep is much richer than that of IVBM, the latter containing only the fully symmetric $U(6)$ irreps (see e.g. [26]). Generally, for non-scalar irreps of $Sp(12, R)$, some of the $U(6)$ irreps belonging to the former may appear several times. Thus a multiplicity index ρ is required, as explicitly shown in (20).

Finally, we note that if we perform a contraction with respect to the index α , then we obtain the many-particle realization of the operators of the one-component nuclear system $F_{ij} = \sum_{\alpha} F_{ij}(\alpha, \alpha)$, $G_{ij} = \sum_{\alpha} G_{ij}(\alpha, \alpha)$ and $A_{ij} = \sum_{\alpha} A_{ij}(\alpha, \alpha)$, which generate the group $Sp(6, R)$. In other words, one obtains the $Sp(6, R)$ model [13] as a submodel of the $Sp(12, R)$ one, in contrast to the microscopic $Sp(12, R)$ model introduced in Ref.[27], in which the components of the mass quadrupole tensor are used as collective variables. Expressing the latter and their derivatives through the boson creation and annihilation operators, among the reduction chains considered in [27],

the three algebraic structures ($U(5)$, $O(6)$ and $SU(3)$) of the IBM-1 [25] were obtained, which are embedded in $Sp(12, R)$ through the group $U(6) \subset Sp(12, R)$ associated with the 6 quadrupole collective degrees of freedom.

V. THE SHELL-MODEL CLASSIFICATION OF NUCLEAR COLLECTIVE STATES

The relevant $Sp(12, R)$ irreducible representations appropriate for the description of the low-lying collective states in heavy mass deformed nuclei - and correspondingly the related $O(m)$ irreps - in the reduction (3) can be fixed by considering the underlying shell-model structure of the ground state. To reveal this structure we specify the basis of an $Sp(12, R)$ irrep by considering the following reduction of the subgroup $U(6) \subset Sp(12, R)$:

$$\begin{aligned} Sp(12, R) &\supset \\ \sigma &\quad n\rho \\ &\supset U(6) \supset SU_p(3) \otimes SU_n(3) \\ &\quad E \quad \gamma \quad (\lambda_p, \mu_p) \quad (\lambda_n, \mu_n) \\ &\supset SU(3) \supset SO(3) \supset SO(2). \\ &\varrho(\lambda, \mu) \quad K \quad L \quad M \end{aligned} \quad (21)$$

The chain (21) naturally generalizes the Elliott's $SU(3)$ model [16] by extending the model space to the direct product space $SU_p(3) \otimes SU_n(3)$ of proton and neutron subsystems. The $SU(3)$ irreps of the two subsystems are subsequently coupled to the $SU(3)$ irrep of the combined proton-neutron system. The combined $SU(3)$ algebra is generated by the quadrupole $Q_M = Q_M^p + Q_M^n$ and angular momentum $L_M = L_M^p + L_M^n$ operators, respectively. The chain (21) corresponds to the following choice of the index $\eta = \gamma(\lambda_p, \mu_p)(\lambda_n, \mu_n)\varrho(\lambda, \mu)KLM$ labeling the basis states (20) of an $Sp(12, R)$ irrep.

The choice of the coupling scheme (21) is dictated by the fact that the dominant component of the nuclear interaction in heavy mass deformed nuclei is provided by the quadrupole-quadrupole forces. The Eq.(21) implies a strong coupling of the proton and neutron distributions to form a composite distribution of the combined proton-neutron system with different possible deformations. The maximum deformation is obtained by restricting the direct product irrep $(\lambda_p, \mu_p) \otimes (\lambda_n, \mu_n)$ of $SU_p(3) \otimes SU_n(3)$ to the leading irreducible representation $(\lambda_p + \lambda_n, \mu_p + \mu_n)$ of $SU(3)$. Then the corresponding geometric picture of the algebraic structure defined by (21) is that of two coupled rotors (one rotor representing the protons and another for the neutrons) [28],[29]. We stress that the reduction of a generic $U(6)$ irreducible representation $E \equiv [E_1, E_2, E_3, E_4, E_5, E_6]$ to the direct product irreps of $SU_p(3) \otimes SU_n(3)$ allows irreps of the type $(\lambda_p, \mu_p) \otimes (\lambda_n, \mu_n)$ with nonzero values of the quantum numbers λ and μ characterizing the proton and neu-

tron $SU(3)$ irreps. The latter geometrically corresponds to two non-axial rotors [29]. This is in contrast to the case of the IVBM [24],[26] in which the model space is spanned only by all fully symmetric $U(6)$ irreps that reduce to the $SU_p(3) \otimes SU_n(3)$ direct product irreps of the type $(\lambda_p, 0) \otimes (\lambda_n, 0)$. Thus, the geometric picture of the latter is that of two coupled axial rotors [29].

The generators of $Sp(12, R)$ (11)–(13) can be classified as irreducible tensor operators with respect to different subgroups of the whole chain (21) and hence will be characterized by the quantum numbers determining their irreducible representations. For the raising operators one readily obtains the following tensors:

$$\begin{aligned} F_{(2,0)(0,0)}^{[2]_6} \quad {}^{LM}_{(2,0)}(p, p), \quad F_{(0,0)(2,0)}^{[2]_6} \quad {}^{LM}_{(2,0)}(n, n), \\ F_{(1,0)(1,0)}^{[2]_6} \quad {}^{LM}_{(2,0)}(p, n), \end{aligned} \quad (22)$$

where $L = 0, 2; M = -L, \dots, M$, and

$$F_{(1,0)(1,0)}^{[2]_6} \quad {}^{1M}_{(0,1)}(p, n). \quad (23)$$

We see a multiplication of the standard one-component $Sp(6, R)$ raising generators [13] which for the two-component system correspond to the creation of monopole and quadrupole pp , nn , and pn pairs. In addition to the $(2, 0)$ $SU(3)$ raising generators $F_{(p,0)(q,0)}^{[2]_6} \quad {}^{LM}_{(2,0)}(\alpha, \beta)$ (22) we have also the $(0, 1)$ $SU(3)$ tensor operator $F_{(1,0)(1,0)}^{[2]_6} \quad {}^{1M}_{(0,1)}(p, n)$ (23), which is a new one compared to the generators of the $Sp(6, R)$ model of Rosensteel and Rowe [13].

The number of bosons operator N is the first Casimir invariant of the $U(6)$ as well as of the combined proton-neutron $U(3)$ group. The latter allows us to determine the shell-model tensor properties by considering the reduction chain $U(3) \supset U(1) \otimes SU(3)$. Thus, in shell-model terms, the raising operators of $Sp(12, R)$ with $U(1) \otimes SU(3)$ quantum numbers $N(\lambda, \mu) = 2(2, 0)$ and their conjugate lowering ones represent $\pm 2\hbar\omega$ inter-shell collective excitations of monopole and quadrupole type. Additionally, in contrast to the $Sp(6, R)$ model, the $Sp(12, R)$ raising operators with $N(\lambda, \mu) = 2(0, 1)$ together with their conjugate lowering operators correspond to the $\pm 2\hbar\omega$ inter-shell excitations of dipole type. Thus, the $Sp(12, R)$ collective dynamics covers the nuclear coherent excitations of monopole, dipole and quadrupole type.

The basis states classified according to (21) can be written as

$$|N_{min}; \sigma n \rho E; \gamma(\lambda_p, \mu_p)(\lambda_n, \mu_n) \varrho(\lambda, \mu); KLM\rangle, \quad (24)$$

where ρ, γ and ϱ are multiplicity indices. Recall that $\sigma = [\sigma_1, \dots, \sigma_6]$, $n = [n_1, \dots, n_6]$, $E = [E_1, \dots, E_6]$. These basis states can be simultaneously classified according to the chain (3). Then the symplectic bandhead structure determined by the $U(6)$ irrep σ will coincide with the $O(m)$ irrep ω , i.e. $\sigma \equiv \omega$ [22].

The appearance of the group $O(m)$ in (3) turns out to be crucial because it allows one to construct the nuclear wave functions with the proper permutational symmetry. The essential property that makes it possible is the fact that the group $O(m)$ contains the symmetric group S_{m+1} as a subgroup. Thus, to fix the permutational symmetry of the wave function, we consider the embedding of the symmetric group S_{m+1} in the $O(m)$ according to

$$\begin{aligned} O(m) \supset S_{m+1}, \\ \omega \quad \delta \quad [f]h \end{aligned} \quad (25)$$

where $[f]$ is the Young scheme characterizing the irreducible representation of the permutational group S_{m+1} , h indexes its basis, and δ is a multiplicity index. However, because the antisymmetry should be satisfied separately for protons and neutrons, in order to insure the proper permutational symmetry we consider further the reduction of S_A to $S_{N_1} \otimes S_{N_2}$ ($A = N_1 + N_2$), i.e.

$$S_A \supset S_{N_1} \otimes S_{N_2}. \quad (26)$$

Taking this into account we replace h by $\delta_0[f_1]h_1[f_2]h_2$, where δ_0 is a multiplicity index in the reduction (26). The full antisymmetry of the total wave function is therefore ensured by coupling of the Young scheme of the S_A irrep to its conjugate (contragradient) representation of the spin wave function.

The basis states classified according to (3), (21) and (25) can be written as

$$|N_{min}; n \rho E; \gamma(\lambda_p, \mu_p)(\lambda_n, \mu_n) \varrho(\lambda, \mu); KLM; \omega \delta[f]h\rangle, \quad (27)$$

where h is a basis of the S_{m+1} irrep $[f]$, which is further fixed by the reduction chain (26). In (24) and (27), N_{min} counts the minimum number of oscillator quanta (phonons) allowed by the Pauli principle.

As we saw, a generic $Sp(12, R)$ irrep is determined by the $U(6)$ lowest weight with $\sigma = [\sigma_1, \dots, \sigma_6]$ and contains all $U(6)$ irreps $E = [E_1, \dots, E_6]$ which are obtained by the $U(6)$ -coupling $[\sigma_1, \dots, \sigma_6] \otimes [n_1, \dots, n_6]$. However, one expects the most symmetric $U(6)$ irreps E represented by the one- and two-rowed Young schemes to be dominant in the low-energy spectra of the heavy deformed even-even nuclei. As an example, the symplectic classification of the $SU(3)$ basis states according to the decompositions given by the chain (21) for the scalar $Sp(12, R)$ irrep $\langle \sigma \rangle = 0$, restricted to the two-rowed $U(6)$ partitions is given in Table II. We see that even for the scalar $Sp(12, R)$ representation one has a very rich algebraic structure of the state space. For comparison, the corresponding $SU(3)$ basis states for the number of oscillator quanta $N = 0, 2, 4, \dots$ contained in the scalar irreducible representation of the $Sp(6, R)$ model [13] are marked in red.

Since the collective states of the $Sp(12, R)$ irreducible spaces for heavy deformed nuclei are constructed from the excitations built on the two adjacent major shell intrinsic structures with opposite parity, then the collective spaces obviously consists of both the positive and

negative parity excitations. From Table II we see the appearance of many new $SU(3)$ multiplets which contain a richer angular momentum and parity content, as well as a multiplication of the $SU(3)$ irreps arising from the coupling of different initial proton and neutron configurations and hence giving rise to distinct coupled proton-neutron $SU(3)$ collective excitations with both the positive and negative parity.

In geometrical terms, from Table II we see that except the two-axial and axial-triaxial, the two-triaxial rotor model configurations, mentioned above, also appear already in the decomposition of the two-rowed $U(6)$ irreps for oscillator quanta $N \geq 6$. It is clear that the $U(6)$ Young schemes with more than two rows will also give rise to the two-triaxial rotor model configurations.

In general, a nonscalar $Sp(12, R)$ irreducible representation $\langle \sigma \rangle \neq 0$ will corresponds to a given real nucleus. This is a very important feature of the present $Sp(12, R)$ collective model, which is a consequence of the two-component composite character of the nuclear systems. A given $Sp(12, R)$ irreducible representation, as was discussed, is determined by the corresponding symplectic bandhead (or intrinsic) structure defined by the lowest $U(6)$ irrep $\sigma = [\sigma_1, \dots, \sigma_6] \neq 0$. The latter, in contrast to the $Sp(6, R)$ case, will contain a plethora of $SU(3)$ multiplets. This is of significant importance in the microscopic nuclear structure theory because the states belonging to the intrinsic space spanned by the $U(6)$ irrep $[\sigma_1, \dots, \sigma_6]$ will contain all the necessary $SU(3)$ irreducible representations needed for the description of different low-lying collective bands (ground state, β , γ , $K^\pi = 0^-, 1^-, 2^-$, etc.) in the spectra of heavy even-even deformed nuclei. In this way, in contrast to the $Sp(6, R)$ model, the intrinsic $Sp(12, R)$ bandhead structure provides us with a framework for the simultaneous shell-model interpretation of the ground state band and the other excited low-lying collective bands without the need of involving the mixing of different symplectic irreps (c.f. Ref.[30]).

How the intrinsic $U(6)$ structure can be determined in practice for a certain nucleus? The proper choice is suggested by the shell model. In this way, for a given nucleus the appropriate symplectic bandhead can be determined by fixing the corresponding underlying proton-neutron shell-model structure $SU_p(3) \otimes SU_n(3) \supset SU(3)$ embedded in the $U(6)$ irrep $[\sigma_1, \dots, \sigma_6]$. The parent $SU(3)$ irreps (λ_p, μ_p) and (λ_n, μ_n) of the two subsystems, which are consequently strongly coupled to the $SU(3)$ irrep (λ, μ) of the combined proton-neutron nuclear system as a whole, are determined by compactly filling pairwise the 3-dimensional harmonic oscillator potential with protons and neutrons, respectively. Then N_{min} in (24) and (27) will counts the total number of oscillator quanta consistent with the Pauli principle, counting all filled levels and remembering to include the factor $\frac{6}{2}m$ for the zero-point motion of the $m = A - 1$ quasi-particles associated with the relative Jacobi vectors, i.e. $N_{min} = (\sigma_1 + \dots + \sigma_6) + \frac{6}{2}m$ [18],[31]-[34].

TABLE II: Symplectic classification of the $SU(3)$ basis states.

N	$[E_1, \dots, E_6]$	(λ_p, μ_p)	(λ_n, μ_n)	(λ, μ)
0	[0]	(0, 0)	(0, 0)	(0, 0)
2	[2]	(2, 0) (1, 0) (0, 0)	(0, 0) (1, 0) (2, 0)	(2, 0) (2, 0), (0, 1) (2, 0)
4	[4]	(4, 0) (3, 0) (2, 0) (1, 0) (0, 0)	(0, 0) (1, 0) (2, 0) (3, 0) (4, 0)	(4, 0) (4, 0), (2, 1) (4, 0), (2, 1), (0, 2) (4, 0), (2, 1) (4, 0)
		(0, 1) (1, 1) (1, 0) (0, 2) (0, 0) (2, 0)	(0, 1) (1, 0) (1, 1) (0, 0) (0, 2) (2, 0)	(0, 2), (1, 0) (2, 1), (0, 2), (1, 0) (2, 1), (0, 2), (1, 0) (0, 2) (0, 2) (4, 0), (2, 1), (0, 2)
	[22]			
6	[6]	(6, 0) (5, 0) (4, 0) (3, 0) (2, 0) (1, 0) (0, 0)	(0, 0) (1, 0) (2, 0) (3, 0) (4, 0) (5, 0) (6, 0)	(6, 0) (6, 0), (4, 1) (6, 0), (4, 1), (2, 2) (6, 0), (4, 1), (2, 2) (6, 0), (4, 1), (2, 2) (6, 0), (4, 1) (6, 0)
		(2, 2) (1, 2) (0, 2) (2, 1) (1, 1) (0, 1) (3, 1) (2, 1) (1, 1) (2, 0) (1, 0) (0, 0) (3, 0) (2, 0) (1, 0) (4, 0) (3, 0) (2, 0)	(0, 0) (1, 0) (2, 0) (0, 1) (1, 1) (2, 1) (1, 0) (2, 0) (3, 0) (0, 2) (1, 2) (2, 2) (1, 1) (2, 1) (3, 1) (2, 0) (3, 0) (4, 0)	(2, 2) (2, 2), (0, 3), (1, 1), (0, 0) (2, 2), (1, 1), (0, 0) (2, 2), (0, 3), (1, 1), (0, 0) (2, 2), 2(1, 1), (0, 0), (3, 0), (0, 3) (2, 2), (3, 0), (1, 1), (0, 0) (4, 1), (2, 2), (0, 3), (3, 0), (0, 1) (4, 1), (2, 2), (0, 3), (3, 0), (0, 1) (4, 1), (2, 2), (0, 3), (3, 0), (0, 1) (2, 2), (1, 1), (0, 0) (2, 2), (0, 3), (1, 1), (0, 0) (2, 2) (4, 1), (2, 2), (0, 3), (3, 0), (0, 1) (4, 1), (2, 2), (0, 3), (3, 0), (0, 1) (4, 1), (2, 2), (0, 3), (3, 0), (0, 1) (6, 0), (4, 1), (2, 2) (6, 0), (4, 1), (2, 2), (0, 3) (6, 0), (4, 1), (2, 2)
	[42]			
...

Having the basis, an arbitrary microscopic or phenomenological Hamiltonian can be diagonalized within the collective symplectic space of $Sp(12, R)$ algebra. The calculation is simplified when the Hamiltonian lies in the enveloping algebra of $Sp(12, R)$ since the latter contains many physically relevant parts of the nuclear forces, like

the proton and neutron harmonic oscillator Hamiltonians, the kinetic energy terms for the two subsystems, the collective potential represented by a scalar function of the full quadrupole operator, and a residual interaction. The latter should include, for example, single-particle spin-orbit and orbit-orbit terms, as well as pairing and other interactions. In practical calculations, however, the Hamiltonian can be restricted to form that is solely expressed in terms of the symplectic generators. Interaction of this form do not mix different symplectic irreps and the Hamiltonian for such interactions will have block-diagonal structure. The single symplectic irrep approximation will be a sensible choice for nuclear systems that have a dominant quadrupole-quadrupole force. The latter favors the states with maximum spatial symmetry and the largest value of the second invariant of the $SU(3)$.

Following the concepts of Refs.[30],[35] we can define the *collective subspaces* (vertical cones) as the irreducible symplectic subspaces of the nuclear Hilbert space. Each $Sp(12, R)$ irrep is characterized by a lowest-weight state with quantum numbers $\sigma = [\sigma_1, \dots, \sigma_6]$. Then if $\{|\sigma\eta\rangle\}$ denotes a basis for the $U(6)$ lowest-weight space of an $Sp(12, R)$ irrep σ , any shell-model state belonging to this collective subspace can be expressed as

$$\psi_\sigma = \sum_\eta \psi_\eta(F) |\sigma\eta\rangle, \quad (28)$$

where ψ_η is a polynomial in the $Sp(12, R)$ raising operators. The lowest-weight state of an $Sp(12, R)$ irrep is referred to as an intrinsic state for that collective subspace. The extension to an arbitrary shell-model state expresses the fact that the shell-model space can be decomposed into a direct sum of $Sp(12, R)$ irreps. Correspondingly, the $Sp(12, R)$ *symplectic shells* (horizontal layers) are defined as the direct sum of all $n\hbar\omega$ states of fixed n , which are obtained by the repeated action of the $Sp(12, R)$ raising operators on the $U(6)$ intrinsic space states. In this way the nuclear Hilbert space naturally divides simultaneously into vertical cones and horizontal layers, reflecting the collective and single-particle aspects of nuclear motion.

We note that the Eq.(28) can be interpreted as a factoring of an arbitrary wave function into collective and intrinsic parts. The states $|\sigma\eta\rangle$ can be thought of as intrinsic states and the raising operators ψ_η as collective wave functions. If the Hamiltonian under consideration consists of terms that mix different symplectic irreps (horizontal mixing), then a sum over σ in the Eq.(28) should be included.

Concluding, we want to point out that other possibilities exist to arrange the low-lying symplectic irreps in the low-lying energy spectra and to fix the intrinsic structure of the ground state. The relevant symplectic bandhead intrinsic structure can be determined by taking into account the proper deformation using the deformed harmonic oscillator (asymptotic Nilsson model [36]) or pseudo- $SU(3)$ [37] schemes of filling the proton and neutron single particle levels.

Finally, in practical calculations, one may use other reductions of the $U(6)$ group (e.g., through the $O(6)$ group appropriate for transitional, γ -unstable, nuclei) to classify the basis states, also consistent with the underlying proton-neutron shell-model structure.

VI. THE SPIN PART

The internal degrees of freedom associated with the group $O(m)$ play an important role in the construction of the microscopic wave functions because they allow one to ensure the full antisymmetry of the total wave function. This is achieved by coupling of the Young scheme of the S_m irrep to its conjugate (contragradient) representation of the spin wave-function.

The construction of the spin function is reduced to the coupling of the two subsystems spins S_1 and S_2 into total spin S of the nucleus [22],[38]:

$$\begin{aligned} & \Phi\left(\widetilde{[f]}\widetilde{\delta_0}\widetilde{[f_1]}\widetilde{h_1}\widetilde{[f_2]}\widetilde{h_2}; SM_S\right) \\ &= \sum_{M_{S_1} M_{S_2}} \Phi\left(\widetilde{[f_1]}\widetilde{h_1}; S_1 M_{S_1}\right) \Phi\left(\widetilde{[f_2]}\widetilde{h_2}; S_2 M_{S_2}\right) \\ & \quad \times C_{M_{S_1} M_{S_2} S}^{S_1 S_2 S}. \end{aligned} \quad (29)$$

The Pauli principle requires the antisymmetry of the wave function with respect to proton and neutron variables separately. That is why only the proton and neutron schemes $[f_1], [\widetilde{f_1}]$ and $[f_2], [\widetilde{f_2}]$ are coupled separately to the antisymmetric irreps a_1 and a_2 , respectively. Then the full antisymmetric wave function can be written in the following form

$$\begin{aligned} & |N_{min}; n\rho E; \gamma(\lambda_p, \mu_p)(\lambda_n, \mu_n)\varrho(\lambda, \mu); K(LS)JM_J; \\ & \omega\delta[f]\delta_0[f_1][f_2]\rangle \\ &= \sum_{h_1 h_2 M M_S} |N_{min}; n\rho E; \gamma(\lambda_p, \mu_p)(\lambda_n, \mu_n)\varrho(\lambda, \mu); \\ & KLM; \omega\delta[f]\delta_0[f_1]h_1[f_2]h_2\rangle \\ & \times \Phi\left(\widetilde{[f]}\widetilde{\delta_0}\widetilde{[f_1]}\widetilde{h_1}\widetilde{[f_2]}\widetilde{h_2}; SM_S\right) C_{h_1 h_1 a_1}^{[f_1] [f_1] a_1} C_{h_2 h_2 a_2}^{[f_2] [f_2] a_2} \\ & \times C_h^{[f]\delta_0} C_{h_1 h_2}^{[f_1] [f_2]} C_M^L C_{M_S}^S C_{M_J}^J, \end{aligned} \quad (30)$$

where $C_{h_1 h_1 a_1}^{[f_1] [f_1] a_1}$, $C_{h_2 h_2 a_2}^{[f_2] [f_2] a_2}$, $C_M^L C_{M_S}^S C_{M_J}^J$ are respectively the Clebsch-Gordan coefficients for the groups S_{N_1} , S_{N_2} and $SU(2)$, and $C_h^{[f]\delta_0} C_{h_1 h_2}^{[f_1] [f_2]}$ is the isoscalar factor for the chain $S_A \supset S_{N_1} \otimes S_{N_2}$.

With this, the task of constructing the fully microscopic antisymmetric wave functions of the proton-neutron $Sp(12, R)$ model of nuclear collective motions is completed. The latter allows the spin contribution of different parts of the nuclear interaction (e.g. spin-orbit, vector, etc. forces) to be included in the consideration, as

well as to encompass the treatment of the odd-mass and odd-odd nuclei together with the even-even ones within a single framework.

VII. CONCLUSIONS

In the present paper, a proton-neutron symplectic model of collective motions, based on the non-compact symplectic group $Sp(12, R)$, is introduced by considering the symplectic geometry of the two-component many-particle nuclear system. The non-compact feature of symplectic scheme is an essential ingredient of the model that allows the theory to accommodate quadrupole coherence which develop in the collective dynamics.

The problem of $6m$ degrees of freedom defined in the many-particle Hilbert space can be associated with a definite irrep of the dynamical symmetry group $Sp(12m, R)$. It was proved, however, that the collective effects are associated with operators that are scalar in $O(m)$. Then, the collective part of the Hamiltonian can be obtained by projecting the latter on a definite $O(m)$ irrep associated with the m Jacobi vectors in the configuration space \mathbb{R}^{6m} , where $m = A - 1$ and A is equal to the total number of nucleons in the system.

From the hydrodynamic perspective, the possible classical collective motions are determined by different dynamical groups that can be constructed from the symplectic generators of $Sp(12, R)$, including a wide class of both the in-phase (isoscalar) and out-of-phase (isovector) excitations of the proton subsystem with respect to the neutron one, as well as collective excitations of the com-

bined proton-neutron system as a whole. The $Sp(12, R)$ group provides therefore a general framework for the investigation of the nature of classical collective motions in nuclei.

The relation of the $Sp(12, R)$ irreps with the shell-model classification of the basis states is considered by extending of the model space to the direct product space of $SU_p(3) \otimes SU_n(3)$ irreps, generalizing in this way the Elliott's $SU(3)$ model for the case of two-component system. The $Sp(12, R)$ model appears then as a natural multi-major-shell extension of the generalized proton-neutron $SU(3)$ scheme which takes into account the core collective excitations of monopole, quadrupole and dipole type associated with the giant vibrational degrees of freedom. Each $Sp(12, R)$ irreducible representation is determined by a symplectic bandhead or an intrinsic $U(6)$ space which can be fixed by the underlying proton-neutron shell-model structure, so the theory becomes completely compatible with the Pauli principle. It is shown that this intrinsic $U(6)$ structure is of vital importance for the appearance of the low-lying collective bands with both the positive and negative parity. The full range of low-lying collective states can then be described by the microscopically based intrinsic $U(6)$ structure, renormalized by coupling to the giant resonance vibrations.

Summarizing, the $Sp(12, R)$ symplectic model provides a natural framework for the simultaneous macroscopic and microscopic description of nuclear collective dynamics of the two-component proton-neutron nuclear systems.

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